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Blow-up phenomena and global existence for the weakly dissipative generalized periodic Degasperis-Procesi equation

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Abstract
In this paper, we investigate the Cauchy problem of a weakly dissipative generalized periodic Degasperis-Procesi equation. The precise blow-up scenarios of strong solutions to the equation are derived by a direct method. Several new criteria guaranteeing the blow-up of strong solutions are presented. The exact blow-up rates of strong solutions are also determined. Finally, we give a new global existence results to the equation.

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1 Introduction

Recently, the following generalized periodic Degasperis-Procesi equation (μDP) was introduced and studied in [1–3]

$$\mu(u)_t - u_{txx} + 3\mu(u)u_x = 3u_x u_{xx} + uu_{xxx},$$

where $u(t,x)$ is a time-dependent function on the unite circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_{\mathbb{S}} u(t,x)\,dx$ denotes its mean. The μDP equation can be formally described as an evolution equation on the space of tensor densities over the Lie algebra of smooth vector fields on the circle $\mathbb{S}$. In [2], the authors verified that the periodic μDP equation describes the geodesic flows of a right-invariant affine connection on the Fréchet Lie group $\mathrm{Diff}^\infty(\mathbb{S})$ of all smooth and orientation-preserving diffeomorphisms of the circle $\mathbb{S}$.

Analogous to the generalized periodic Camassa-Holm (μCH) equation [4–6], μDP equation possesses bi-Hamiltonian form and infinitely many conservation laws. Here we list some of the simplest conserved quantities:

$$H_0 = -\frac{9}{2}\int_{\mathbb{S}} y\,dx, \qquad H_1 = \frac{1}{2}\int_{\mathbb{S}} u^2\,dx, \qquad H_2 = \int_{\mathbb{S}}\left(\frac{3}{2}\mu(u)\left(A^{-1}\partial_x u\right)^2 + \frac{1}{6}u^3\right)dx,$$

where $y = \mu(u) - u_{xx}$, $A = \mu - \partial_x^2$ is an isomorphism between H^S and H^{s-1}. Moreover, it is easy to see that $\int_{\mathbb{S}} u(t,x)\,dx$ is also a conserved quantity for the μDP equation.

Obviously, under the constraint of $\mu \equiv 0$, the μDP equation is reduced to the μBurgers equation [7].

It is clear that the closest relatives of the μDP equation are the DP equation [8–11]

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx},$$

which was derived by Degasperis and Procesi in [8] as a model for the motion of shallow water waves, and its asymptotic accuracy is the same as for the Camassa-Holm equation.

Generally speaking, energy dissipation is a very common phenomenon in the real world. It is interesting for us to study this kind of equation. Recently, Wu and Yin [12] considered the weakly dissipative Degasperis-Procesi equation. For related studies, we refer to [13] and [14]. Liu and Yin [15] discussed the blow-up, global existence for the weakly dissipative μ-Hunter-Saxton equation.

In this paper, we investigate the Cauchy problem of the following weakly dissipative periodic Degasperis-Procesi equation [16]:

$$\begin{cases} \mu(u)_t - u_{txx} + 3\mu(u)u_x = 3u_x u_{xx} + uu_{xxx} - \lambda(\mu(u) - u_{xx}), & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}, \end{cases} \tag{1.1}$$

the constant λ is a nonnegative dissipative parameter and the term $\lambda y = \lambda(\mu(u) - u_{xx})$ models energy dissipation. Obviously, if $\lambda = 0$ then the equation reduces to the μDP equation. we can rewrite the system (1.1) as follows:

$$\begin{cases} y_t + uy_x + 3u_x y + \lambda y = 0, & t > 0, x \in \mathbb{R}, \\ y = \mu(u) - u_{xx}, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}. \end{cases} \tag{1.2}$$

Let $G(x) := \frac{1}{2}x^2 - \frac{1}{2}|x| + \frac{13}{12}$, $x \in \mathbb{R}$ be the associated Green's function of the operator A^{-1}, then the operator can be expressed by its associated Green's function,

$$A^{-1}f(x) = (G * f)(x), \quad f \in L^2,$$

where $*$ denotes the spatial convolution. Then equation (1.1) takes the equivalent form of a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} u_t + uu_x + 3\mu(u)A^{-1}\partial_x u + \lambda u = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}. \end{cases} \tag{1.3}$$

It is easy to check that the operator $A = \mu - \partial_x^2$ has the inverse

$$\begin{aligned} \left(A^{-1}f\right)(x) = {} & \left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{13}{12}\right)\mu(f) + \left(x - \frac{1}{2}\right)\int_0^1 \int_0^y f(s)\,ds\,dy \\ & - \int_0^x \int_0^y f(s)\,ds\,dy + \int_0^1 \int_0^y \int_0^s f(r)\,dr\,ds\,dy. \end{aligned} \tag{1.4}$$

Since A^{-1} and ∂_x commute, the following identities hold:

$$\big(A^{-1}\partial_x f\big)(x) = \left(x - \frac{1}{2}\right)\int_0^1 f(x)\,dx - \int_0^x f(y)\,dy + \int_0^1 \int_0^x f(y)\,dy\,dx \tag{1.5}$$

and

$$\big(A^{-1}\partial_x^2 f\big)(x) = -f(x) + \int_0^1 f(x)\,dx. \tag{1.6}$$

The paper is organized as follows. In Section 2, we briefly give some needed results, including the local well-posedness of equation (1.1), and some useful lemmas and results which will be used in subsequent sections. In Section 3, we establish the precise blow-up scenarios and blow-up criteria of strong solutions. In Section 4, we give the blow-up rate of strong solutions. In Section 5, we give two global existence results of strong solutions.

Remark 1.1 Although blow-up criteria and global existence results of strong solutions to equation (1.1) are presented in [16], our blow-up results improve considerably earlier results.

2 Preliminaries

In this section we recall some elementary results which we want to use in this paper. We list them and skip their proofs for conciseness. Local well-posedness for equation (1.1) can be obtained by Kato's theory [17], in [16] the authors gave a detailed description on well-posedness theorem.

Theorem 2.1 [16] *Let $s > 3/2$ and $u_0 \in H^s(\mathbb{S})$; then there is a maximal time T and a unique solution*

$$u \in C\big([0,T); H^s(\mathbb{S})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{S})\big)$$

of the Cauchy problems (1.1) *which depends continuously on the initial data*, *i.e. the mapping*

$$H^s(\mathbb{S}) \to C\big([0,T); H^s(\mathbb{S})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{S})\big), \quad u_0 \mapsto u(\cdot, u_0),$$

is continuous.

Remark 2.1 The maximal time of existence $T > 0$ in Theorem 2.1 is independent of the Sobolev index $s > 3/2$.

Next we present the Sobolev-type inequalities, which play a key role to obtain blow-up results for the Cauchy problem (1.1) in the sequel.

Lemma 2.2 [18] *If $f \in H^1(\mathbb{S})$ is such that $\int_{\mathbb{S}} f(x)\,dx = 0$, then we have*

$$\max_{x\in\mathbb{S}} f^2(x) \le \frac{1}{12}\int_{\mathbb{S}} f_x^2(x)\,dx.$$

Lemma 2.3 [19] *If* $r > 0$, *let* $\Lambda = (1-\partial_x^2)^{1/2}$, *then*

$$\big\|[\Lambda^r, f]g\big\|_{L^2} \le c\big(\|\partial_x f\|_{L^\infty}\big\|\Lambda^{r-1}g\big\|_{L^2} + \big\|\Lambda^r f\big\|_{L^2}\|g\|_{L^\infty}\big),$$

where c *is a constant depending only on* r.

Lemma 2.4 [20] *Let* $t_0 > 0$ *and* $v \in C^1([0,t_0); H^2(\mathbb{R}))$, *then for every* $t \in [0,t_0)$ *there exists at least one point* $\xi(t) \in \mathbb{R}$ *with*

$$m(t) := \inf_{x\in\mathbb{R}} v_x(t,x) = v_x\big(t,\xi(t)\big),$$

and the function m *is almost everywhere differentiable on* $(0,t_0)$ *with*

$$\frac{d}{dt}m(t) = v_{tx}\big(t,\xi(t)\big) \quad \text{a.e. on } (0,t_0).$$

We also need to introduce the classical particle trajectory method which is motivated by McKean's deep observation for the Camassa-Holm equation in [21]. Suppose $u(x,t)$ is the solution of the Camassa-Holm equation and $q(x,t)$ satisfies the following equation:

$$\begin{cases} q_t = u(q,t), & 0 < t < T, x \in \mathbb{R}, \\ q(x,0) = x, & x \in \mathbb{R}, \\ q(x+1,t) = x, & 0 < t < T, x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where T is the maximal existence time of solution, then $q(t,\cdot)$ is a diffeomorphism of the line. Taking the derivative with respect to x, we have

$$\frac{dq_x}{dt} = q_{tx} = u_x(q,t)q_x, \quad t \in (0,T).$$

Hence

$$q_x(x,t) = \exp\left(\int_0^t u_x(q,s)\,ds\right) > 0, \qquad q_x(x,0) = 1, \tag{2.2}$$

which is always positive before the blow-up time.

In addition, integrating both sides of the first equation in equation (1.1) with respect to x on $\mathbb{S}$, we obtain

$$\frac{d}{dt}\mu(u) = -\lambda\mu(u),$$

it follows that

$$\mu(u) = \mu(u_0)e^{-\lambda t} := \mu_0 e^{-\lambda t}, \tag{2.3}$$

where

$$\mu_0 := \mu(u_0) = \int_{\mathbb{S}} u_0(x)\,dx. \tag{2.4}$$

3 Blow-up solutions

In this section, we are able to derive an import estimate for the L^∞-norm of strong solutions. This enables us to establish precise blow-up scenario and several blow-up results for equation (1.1).

Lemma 3.1 *Let $u_0 \in H^s$, $s > 3/2$ be given and assume the T is the maximal existence time of the corresponding solution u to equation* (1.1) *with the initial data u_0. Then we have*

$$\|u(t,x)\|_{L^\infty} \le e^{-\lambda t}\left(\frac{3|\mu_0|(\frac{1}{2}|\mu_0| + 2\mu_2)}{\lambda} + \|u_0\|_{L^\infty}\right), \quad \forall t \in [0,T). \tag{3.1}$$

Proof The first equation of the Cauchy problem (1.1) is

$$u_t + uu_x + 3\mu(u)A^{-1}\partial_x u + \lambda u = 0.$$

In view of equation (1.5), we have

$$|A^{-1}\partial_x u| \le \frac{1}{2}|\mu_0|e^{-\lambda t} + 2\left(\int_{\mathbb{S}} u^2\,dx\right)^{\frac{1}{2}}.$$

A direct computation implies that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} u^2\,dx &= 2\int_{\mathbb{S}} 2uu_t\,dx \\ &= -2\int_{\mathbb{S}} 2u\big(uu_x + 3\mu(u)A^{-1}\partial_x u + \lambda u\big)\,dx \\ &= -2\lambda\int_{\mathbb{S}} u^2\,dx.\end{aligned}$$

It follows that

$$\int_{\mathbb{S}} u^2\,dx = \int_{\mathbb{S}} u_0^2\,dx \cdot e^{-2\lambda t} := \mu_2^2 e^{-2\lambda t}. \tag{3.2}$$

So we have

$$|A^{-1}\partial_x(u)| \le \left(\frac{1}{2}|\mu_0| + 2\mu_2\right)e^{-\lambda t}.$$

In view of equation (2.1) we have

$$\frac{du(t,q(t,x))}{dt} = u_t\big(t,q(t,x)\big) + u_x\big(t,q(t,x)\big)\frac{dq(t,x)}{dt} = (u_t + uu_x)\big(t,q(t,x)\big).$$

Combing the above relations, we arrive at

$$\left|\frac{du(t,q(t,x))}{dt} + \lambda u\big(t,q(t,x)\big)\right| \le 3|\mu_0|\left(\frac{1}{2}|\mu_0| + 2\mu_2\right)e^{-2\lambda t}.$$

Integrating the above inequality with respect to $t < T$ on $[0,t]$ yields

$$\left|e^{\lambda t}u\big(t,q(t,x)\big) - u_0(x)\right| \le \frac{3|\mu_0|(\frac{1}{2}|\mu_0| + 2\mu_2)}{\lambda}.$$

Thus

$$\left|u\big(t,q(t,x)\big)\right| \le \left\|u\big(t,q(t,x)\big)\right\|_{L^\infty} \le e^{-\lambda t}\left(\frac{3|\mu_0|(\frac{1}{2}|\mu_0|+2\mu_2)}{\lambda}+\|u_0\|_{L^\infty}\right).$$

In view of the diffeomorphism property of $q(t,\cdot)$, we can obtain

$$\left|u(t,x)\right| \le \left\|u(t,x)\right\|_{L^\infty} \le e^{-\lambda t}\left(\frac{3|\mu_0|(\frac{1}{2}|\mu_0|+2\mu_2)}{\lambda}+\|u_0\|_{L^\infty}\right).$$

This completes the proof of Lemma 3.1. □

Theorem 3.2 *Let $u_0 \in H^s$, $s > 3/2$ be given and assume that T is the maximal existence time of the corresponding solution $u(t,x)$ to the Cauchy problem* (1.1) *with the initial data u_0. If there exists $M > 0$ such that*

$$\left\|u_x(t,\cdot)\right\|_{L^\infty} \le M, \quad t \in [0,T),$$

then the H^s-norm of $u(t,\cdot)$ does not blow up on $[0,T)$.

Proof We assume that c is a generic positive constant depending only on s. Let $\Lambda = (1-\partial_x^2)^{1/2}$. Applying the operator Λ^s to the first one in equation (1.3), multiplying by $\Lambda^s u$, and integrating over $\mathbb{S}$, we obtain

$$\frac{d}{dt}\|u\|_{H^s}^2 = -2(uu_x,u)_{H^s} - 6\big(u,A^{-1}\partial_x\big(\mu(u)u\big)\big)_{H^s} - 2\lambda(u,u)_{H^s}. \tag{3.3}$$

Let us estimate the first term of the above equation,

$$\begin{aligned}
\left|(uu_x,u)_{H^s}\right| &= \left|\big(\Lambda^s(uu_x),\Lambda^s u\big)_{L^2}\right| = \left|\big(\big[\Lambda^s,u\big]u_x,\Lambda^s u\big)_{L^2} + \big(u\Lambda^s u_x,\Lambda^s u\big)_{L^2}\right| \\
&\le \left\|\big[\Lambda^s,u\big]u_x\right\|_{L^2}\left\|\Lambda^s u\right\|_{L^2} + \frac{1}{2}\left|\big(u_x\Lambda^s u,\Lambda^s u\big)_{L^2}\right| \\
&\le 2\left\|(u,v)\right\|_{H^1\times H^1}^2\big(2\left\|(u,v)\right\|_{H^1\times H^1}^2\big) \\
&\le c\|u_x\|_{L^\infty}\|u\|_{H^s}^2,
\end{aligned} \tag{3.4}$$

where we used Lemma 2.3 with $r = s$. Furthermore, we estimate the second term of the right hand side of equation (3.3) in the following way:

$$\begin{aligned}
\left|\big(u,A^{-1}\partial_x\big(\mu(u)u\big)\big)_{H^s}\right| &= \left|\big(u,A^{-1}\partial_x\big(e^{-\lambda t}\mu_0 u\big)\big)_{H^s}\right| \\
&\le e^{-\lambda t}|\mu_0|\|u\|_{H^s}\left\|A^{-1}\partial_x u\right\|_{H^s} \\
&\le c|\mu_0|\|u\|_{H^s}^2.
\end{aligned} \tag{3.5}$$

Combing equations (3.4) and (3.5) with equation (3.3) we arrive at

$$\frac{d}{dt}\|u\|_{H^s}^2 \le c\big(|\mu_0| + \|u_x\|_{L^\infty} + 2\lambda\big)|\|u\|_{H^s}^2.$$

An application of Gronwall's inequality and the assumption of the theorem yield

$$\|u\|_{H^s}^2 \le e^{c(|\mu_0|+M+2\lambda)t}\|u_0\|_{H^s}^2.$$

This completes the proof of the theorem. □

The following result describes the precise blow-up scenario. Although the result which is proved in [16], our method is new, concise, and direct.

Theorem 3.3 *Let $u_0 \in H^s$, $s > 3/2$ be given and assume that T is the maximal existence time of the corresponding solution $u(t,x)$ to the Cauchy problem* (1.1) *with the initial data u_0. Then the corresponding solution blows up in finite time if and only if*

$$\liminf_{t\to T}\Big\{\inf_{x\in\mathbb{S}} u_x(t,x)\Big\} = -\infty.$$

Proof Since the maximal existence time T is independent of the choice of s by Theorem 2.1, applying a simple density argument, we only need to consider the case $s = 3$. Multiplying the first one in equation (1.2) by y and integrating over $\mathbb{S}$ with respect to x yield

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} y^2\,dx &= 2\int_{\mathbb{S}} yy_t\,dx = -2\int_{\mathbb{S}} y(uy_x + 3u_xy + \lambda y)\,dx \\ &= -2\int_{\mathbb{S}} uyy_x\,dx - 6\int_{\mathbb{S}} u_xy^2\,dx - 2\lambda\int_{\mathbb{S}} y^2\,dx \\ &= -5\int_{\mathbb{S}} u_xy^2\,dx - 2\lambda\int_{\mathbb{S}} y^2\,dx.\end{aligned}$$

If u_x is bounded from below on $[0,T)\times\mathbb{S}$, then there exists $N > \lambda > 0$ such that

$$u_x(t,x) \ge -N, \quad \forall (t,x)\in[0,T)\times\mathbb{S},$$

then

$$\frac{d}{dt}\int_{\mathbb{S}} y^2\,dx \le (5N-2\lambda)\int_{\mathbb{S}} y^2\,dx.$$

Applying Gronwall's inequality then yields for $t\in[0,T)$

$$\int_{\mathbb{S}} y^2\,dx \le e^{(5N-2\lambda)t}\int_{\mathbb{S}} y^2(0,x)\,dx.$$

Note that

$$\int_{\mathbb{S}} y^2\,dx = \mu^2(u) + \int_{\mathbb{S}} u_{xx}^2\,dx \ge \|u_{xx}\|_{L^2}^2.$$

Since $u_x \in H^2 \subset H^1$ and $\int_{\mathbb{S}} u_x = 0$, Lemma 2.2 implies that

$$\|u_x\|_{L^\infty} \le \frac{1}{2\sqrt{3}}\|u_{xx}\|_{L^2} \le e^{\frac{(5N-2\lambda)t}{2}}\big\|y(0,x)\big\|_{L^2}.$$

Theorem 3.1 ensures that the solution u does not blow up in finite time. On the other hand, by the Sobolev embedding theorem it is clear that if

$$\liminf_{t\to T}\Big\{\inf_{x\in\mathbb{S}} u_x(t,x)\Big\} = -\infty,$$

then $T < \infty$. This completes the proof of the theorem. □

We now give first sufficient conditions to guarantee wave breaking.

Theorem 3.4 *Let $u_0 \in H^s$, $s > 3/2$ and T be the maximal time of the solution $u(t,x)$ to equation* (1.1) *with the initial data u_0. If*

$$\inf_{x\in\mathbb{S}} u_0'(x) < -\frac{1}{2}\lambda - \frac{1}{2}\sqrt{\lambda^2 + 4\alpha},$$

then the corresponding solution to equation (1.1) *blow up in finite time in the following sense*: *there exists T_0 satisfying*

$$0 < T_0 \le \frac{1}{\sqrt{\lambda^2+4\alpha}} \ln\left(\frac{2\inf_{x\in\mathbb{S}} u_0'(x) + \lambda - \sqrt{\lambda^2+4\alpha}}{2\inf_{x\in\mathbb{S}} u_0'(x) + \lambda + \sqrt{\lambda^2+4\alpha}}\right),$$

where $\alpha = 3|\mu_0|(\frac{3|\mu_0|(\frac{1}{2}|\mu_0|+2\mu_2)}{\lambda} + \|u_0\|_{L^\infty})$, such that

$$\liminf_{t\to T_0}\Big\{\inf_{x\in\mathbb{S}} u_x(t,x)\Big\} = -\infty.$$

Proof As mentioned early, we only need to consider the case $s = 3$. Let

$$m(t) := \inf_{x\in\mathbb{S}}\big[u_x(t,x)\big], \quad t \in [0,T)$$

and let $\xi(t) \in \mathbb{S}$ be a point where this minimum is attained by using Lemma 2.4. It follows that

$$m(t) = u_x\big(t,\xi(t)\big).$$

Differentiating the first one in equation (1.3) with respect to x, we have

$$u_{tx} + u_x^2 + uu_{xx} + 3\mu(u)A^{-1}\partial_x^2 u + \lambda u_x = 0.$$

From equation (1.6) we deduce that

$$u_{tx} = -u_x^2 - uu_{xx} + 3\mu(u)(u-\mu_0) - \lambda u_x. \tag{3.6}$$

Obviously $u_{xx}(t,\xi(t)) = 0$ and $u(t,\cdot) \in H^3(\mathbb{S}) \subset C^2(\mathbb{S})$. Substituting $(t,\xi(t))$ into equation (3.6), we get

$$\begin{aligned}\frac{dm(t)}{dt} &= -m^2(t) - \lambda m(t) + 3\mu(u)u\bigl(t,\xi(t)\bigr) - 3\mu^2(u) \\ &= -m^2(t) - \lambda m(t) + 3\mu_0 e^{-\lambda t}u\bigl(t,\xi(t)\bigr) - 3\mu_0^2 e^{-2\lambda t} \\ &\leq -m^2(t) - \lambda m(t) + 3|\mu_0|\biggl(\frac{3|\mu_0|(\frac{1}{2}|\mu_0| + 2\mu_2)}{\lambda} + \|u_0\|_{L^\infty}\biggr).\end{aligned}$$

Set

$$\alpha = 3|\mu_0|\biggl(\frac{3|\mu_0|(\frac{1}{2}|\mu_0| + 2\mu_2)}{\lambda} + \|u_0\|_{L^\infty}\biggr).$$

Then we obtain

$$\begin{aligned}\frac{dm(t)}{dt} &\leq -m^2(t) - \lambda m(t) + \alpha \\ &\leq -\frac{1}{4}\bigl(2m(t) + \lambda + \sqrt{\lambda^2 + 4\alpha}\bigr)\bigl(2m(t) + \lambda - \sqrt{\lambda^2 + 4\alpha}\bigr).\end{aligned}$$

Note that if $m(0) < -\frac{1}{2}\lambda - \frac{1}{2}\sqrt{\lambda^2 + 4\alpha}$, then $m(t) < -\frac{1}{2}\lambda - \frac{1}{2}\sqrt{\lambda^2 + 4\alpha}$ for all $t \in [0,T)$. From the above inequality we obtain

$$\frac{2m(0) + \lambda + \sqrt{\lambda^2 + 4\alpha}}{2m(0) + \lambda - \sqrt{\lambda^2 + 4\alpha}} e^{\sqrt{\lambda^2+4\alpha}t} - 1 \leq \frac{2\sqrt{\lambda^2 + 4\alpha}}{2m(t) + \lambda - \sqrt{\lambda^2 + 4\alpha}} \leq 0.$$

Since

$$0 < \frac{2m(0) + \lambda + \sqrt{\lambda^2 + 4\alpha}}{2m(0) + \lambda - \sqrt{\lambda^2 + 4\alpha}} < 1,$$

then there exists T_0,

$$0 < T_0 \leq \frac{1}{\sqrt{\lambda^2 + 4\alpha}} \ln\biggl(\frac{2m(0) + \lambda - \sqrt{\lambda^2 + 4\alpha}}{2m(0) + \lambda + \sqrt{\lambda^2 + 4\alpha}}\biggr)$$

such that $\lim_{t\to T_0} m(t) = -\infty$. Theorem 3.3 implies that the solution u blows up in finite time. □

We give another blow-up result for the solutions of equation (1.1).

Theorem 3.5 *Let $u_0 \in H^s$, $s > 3/2$ and T be the maximal time of the solution $u(t,x)$ to equation* (1.1) *with the initial data u_0. If u_0 is odd satisfies $u_0' < -\lambda$, then the corresponding solution to equation* (1.1) *blows up in finite time.*

Proof By $\mu(u(t,-x)) = \mu_0(t,-x)e^{-\lambda t} = -\mu_0(t,x)e^{-\lambda t} = -\mu(u(t,x))$, we can check the function

$$v(t,x) := -u(t,-x), \quad t \in [0,T), x \in \mathbb{R},$$

is also a solution of equation (1.1), therefore $u(x,t)$ is odd for any $t \in [0,T)$. By continuity with respect to x of u and u_{xx}, we get

$$u(t,0) = u_{xx}(t,0) = 0, \quad \forall t \in [0,T).$$

Define $h(t) := u_x(t,0)$ for $t \in [0,T)$. From equation (3.6), we obtain

$$\begin{aligned}\frac{dh(t)}{dt} &= -h^2(t) - \lambda h(t) - 3\mu^2(u)\\ &\le -h^2(t) - \lambda h(t)\\ &= -h(t)\bigl(h(t) + \lambda\bigr).\end{aligned}$$

Note that if $h(0) < -\lambda$, then $h(t) < -\lambda$ for all $t \in [0,T)$. From the above inequality we obtain

$$\left(1 + \frac{\lambda}{h(0)}\right)e^{\lambda t} - 1 \le \frac{\lambda}{h(t)} \le 0.$$

Since

$$0 < \frac{h(0) + \lambda}{h(0)} < 1,$$

there exists T_0,

$$0 < T_0 \le \frac{1}{\lambda} \ln \frac{h(0)}{h(0) + \lambda}$$

such that $\lim_{t \to T_0} m(t) = -\infty$. Theorem 3.3 implies that the solution u blows up in finite time. □

4 Blow-up rate

In this section, we consider the blow-up profile; the blow-up rate of equation (1.1) with respect to time can be shown as follows.

Theorem 4.1 *Let $u_0 \in H^s$, $s > 3/2$ and T be the maximal time of the solution $u(t,x)$ to equation* (1.1) *with the initial data u_0. If T is finite, then*

$$\lim_{t \to T}\Bigl\{(T - t)\min_{x \in \mathbb{S}} u_x(x,t)\Bigr\} = -1.$$

Proof It is inferred from Lemma 2.4 that the function

$$m(t) := \min_{x \in \mathbb{S}} u_x(x,t) = u_x\bigl(t,\xi(t)\bigr)$$

is locally Lipschitz with $m(t) < 0$, $t \in [0,T)$. Note that $u_{xx} = 0$, a.e. $t \in [0,T)$. Then we deduce that

$$\begin{aligned}\bigl|m'(t) + m^2(t) + \lambda m(t)\bigr| &= \bigl|3\mu(u)u\bigl(t,\xi(t)\bigr) - 3\mu^2(u)\bigr|\\ &= \bigl|3\mu_0 e^{-\lambda t}u\bigl(t,\xi(t)\bigr) - 3\mu_0^2 e^{-2\lambda t}\bigr|\\ &\le 3|\mu_0|\left(\frac{3|\mu_0|(\frac{1}{2}|\mu_0| + 2\mu_2)}{\lambda} + \|u_0\|_{L^\infty} + |\mu_0|\right) := K.\end{aligned}$$

It follows that

$$-K \le m'(t) + m^2(t) + \lambda m(t) \le K \quad \text{a.e. on } (0,T). \tag{4.1}$$

Thus,

$$-K - \frac{1}{4}\lambda^2 \le m'(t) + \left(m(t) + \frac{1}{2}\lambda\right)^2 \le K + \frac{1}{4}\lambda^2 \quad \text{a.e. on } (0,T).$$

Now fix any $\varepsilon \in (0,1)$. In view of Theorem 3.1, there exists $t_0 \in (0,T)$ such that $m(t_0) < -\sqrt{(K + \frac{1}{4}\lambda^2)(1 + \frac{1}{\varepsilon})} - \frac{1}{2}\lambda$. Being locally Lipschitz, the function $m(t)$ is absolutely continuous on $[0,T)$. It then follows from the above inequality that $m(t)$ is decreasing on $[t_0,T)$ and satisfies

$$m(t) < -\sqrt{\left(K + \frac{1}{4}\lambda^2\right)\left(1 + \frac{1}{\varepsilon}\right)} - \frac{1}{2}\lambda, \quad t \in [t_0,T).$$

Since $m(t)$ is decreasing on $[t_0,T)$, it follows that

$$\lim_{t\to T} m(t) = -\infty.$$

It is found from equation (4.1) that

$$1 - \varepsilon \le \frac{d}{dt}\left(m(t) + \frac{1}{2}\lambda\right)^{-1} = -\frac{m'(t)}{(m(t) + \frac{1}{2}\lambda)^2} \le 1 + \varepsilon. \tag{4.2}$$

Integrating both sides of equation (4.2) on (t,T), we obtain

$$(1-\varepsilon)(T-t) \le -\frac{1}{(m(t) + \frac{1}{2}\lambda)} \le (1+\varepsilon)(T-t), \quad t \in [t_0,T), \tag{4.3}$$

that is,

$$\frac{1}{(1+\varepsilon)} - \le \left(m(t) + \frac{1}{2}\lambda\right)(T-t) \le \frac{1}{(1-\varepsilon)}, \quad t \in [t_0,T). \tag{4.4}$$

By the arbitrariness of $\varepsilon \in (0, \frac{1}{2})$, we have

$$\lim_{t\to T} (T-t)\bigl(m(t) + \lambda\bigr) = -1. \tag{4.5}$$

This completes the proof of the theorem. □

5 Global existence

In this section, we will present some global existence results. Let us now prove the following lemma.

Lemma 5.1 *Let $u_0 \in H^s$, $s > 3/2$ be given and assume that $T > 0$ is the maximal existence time of the corresponding solution $u(t,x)$ to the Cauchy problem* (1.1). *Let $q \in C^1([0,T) \times$

$\mathbb{R};\mathbb{R})$ *be the unique solution of equation* (2.1). *Then we have*

$$y\big(t,q(t,x)\big)q_x^3 = y_0(x)e^{-\lambda t},$$

where $y = \mu(u) - u_{xx}$.

Proof By the first one in equation (1.2) and equation (2.1) we have

$$\begin{aligned}
\frac{d}{dt}y\big(t,q(t,x)\big)q_x^3 &= (y_t + y_x q_t)q_x^3 + 3yq_xq_{xt} \\
&= (y_t + y_x u)q_x^3 + 3yq_xq_{xt} \\
&= (y_t + uy_x + 3yu_xy_xu)q_x^3 \\
&= -\lambda yq_x^3.
\end{aligned}$$

Therefore

$$y\big(t,q(t,x)\big)q_x^3 = y_0(x)e^{-\lambda t}. \qquad \square$$

Lemma 5.1 and equation (2.2) imply that y and y_0 have the same sign.

Theorem 5.2 *Let* $u_0 \in H^s$, $s > 3/2$. *If* $y_0 = \mu_0 - u_{0,xx} \in H^1$ *does not change sign, then the corresponding solution* $u(t,x)$ *to equation* (1.1) *with the initial data* u_0 *exists globally in time.*

Proof By equation (2.1), we know that $q(t,\cdot)$ is diffeomorphism of the line and the periodicity of u with respect to spatial variable x, given $t \in [0,T)$, there exists a $\xi(t) \in \mathbb{S}$ such that $u_x(t,\xi(t)) = 0$.

We first consider the case that $y_0 \geq 0$ on $\mathbb{S}$, in which case Lemma 5.1 ensures that $y \geq 0$. For $x \in [\xi(t),\xi(t)+1]$, we have

$$\begin{aligned}
-u_x(t,x) &= -\int_{\xi(t)}^x u_{xx}(t,x)\,dx = \int_{\xi(t)}^x \big(y - \mu(u)\big)\,dx \\
&= \int_{\xi(t)}^x y\,dx - \mu(u)\big(x-\xi(t)\big) \leq \int_{\mathbb{S}} y\,dx - \mu(u)\big(x-\xi(t)\big) \\
&= \mu(u)\big(1 - x + \xi(t)\big) \leq |\mu_0|.
\end{aligned}$$

It follows that $u_x(t,x) \geq -|\mu_0|$.

On the other hand, if $y_0 \leq 0$ on $\mathbb{S}$, then Lemma 5.1 ensures that $y \leq 0$. Therefore, for $x \in [\xi(t),\xi(t)+1]$, we have

$$\begin{aligned}
-u_x(t,x) &= -\int_{\xi(t)}^x u_{xx}(t,x)\,dx = \int_{\xi(t)}^x \big(y - \mu(u)\big)\,dx \\
&= \int_{\xi(t)}^x y\,dx - \mu(u)\big(x-\xi(t)\big) \\
&\leq -\mu(u)\big(x-\xi(t)\big) \leq |\mu_0|.
\end{aligned}$$

It follows that $u_x(t,x) \geq -|\mu_0|$. By using Theorem 3.2, we immediately conclude that the solution is global. This completes the proof of the theorem. □

Corollary 5.3 *If the initial value $u_0 \in H^3$ such that*

$$\|\partial_x^3 u_0\|_{L^2} \leq 2\sqrt{3}|\mu_0|,$$

then the corresponding solution u of the initial value u_0 exists globally in time.

Proof Since $\int_{\mathbb{S}} \partial_x^2 u_0 \, dx = 0$, by Lemma 2.2, we obtain

$$\|\partial_x^2 u_0\|_{L^\infty} \leq \frac{1}{2\sqrt{3}} \|\partial_x^3 u_0\|_{L^2}.$$

If $\mu_0 \geq 0$, we have

$$y_0 = \mu_0 - \partial_x^2 u_0 \geq \mu_0 - \frac{1}{2\sqrt{3}} \|\partial_x^3 u_0\|_{L^2} \geq \mu_0 - |\mu_0| = 0.$$

If $\mu_0 < 0$, we have

$$y_0 = \mu_0 - \partial_x^2 u_0 \leq \mu_0 + \|\partial_x^2 u_0\|_{L^\infty} \leq \mu_0 + \frac{1}{2\sqrt{3}} \|\partial_x^3 u_0\|_{L^2} \leq \mu_0 + |\mu_0| = 0. \qquad \square$$

Thus the theorem is proved by using Theorem 5.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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